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LETTER TO THE EDITOR

Deriving *N*-soliton solutions via constrained flows

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Abstract. The soliton equations can be factorized by two commuting x- and t-constrained flows. We propose a method to derive N-soliton solutions of soliton equations directly from the x- and t-constrained flows.

1. Introduction

In recent years much work has been devoted to the constrained flows of soliton equations (see, for example, [1–7]). It was shown in [1–3] that the (1 + 1)-dimensional soliton equation can be factorized by an x- and a t-constrained flow which can be transformed into two commuting x- and t-finite-dimensional integrable Hamiltonian systems. The Lax representation for constrained flows can be deduced from the adjoint representation of the auxiliary linear problem for soliton equations [4]. By means of the Lax representation, the standard method in [8–10] enables us to introduce the separation variables for constrained flows [11–15] and to establish the Jacobi inversion problem [13–15]. Finally, the factorization of soliton equations and the separability of the constrained flows allow us to find the Jacobi inversion problem for soliton equations [13–15]. By using the Jacobi inversion technique [16, 17], the N-gap solutions can be obtained in terms of Riemann theta functions for soliton equations; namely, the constrained flows can be used to derive the N-gap solution for soliton equations. It is believed that the constrained flows can also been used directly to derive the N-soliton solutions for soliton equations. However, this case remains a challenging problem.

It is well known that there are several methods to derive the N-soliton solution of soliton equations, such as the inverse scattering method, the Hirota method, the dressing method, the Darboux transformation, etc (see, for example, [18–20] and references therein). Here, we propose a method to construct directly an N-soliton solution from two commuting x- and t-constrained flows. We will illustrate the method by the KdV equation. The method can be applied to other soliton equations.

2. Constrained flows

We first recall the constrained flows and factorization of soliton equations by using the KdV equation. Let us consider the Schrödinger spectral problem

$$-\phi_{xx} + u\phi = \lambda\phi. \qquad (2.1)$$

The KdV hierarchy associated with (2.1) can be written in infinite-dimensional integrable Hamiltonian system [18–20]

$$u_{t_n} = \partial_x \frac{\delta H_n}{\delta u} \qquad n = 1, 2, \ldots \tag{2.2}$$

where

$$\frac{\delta H_n}{\delta u} = L^n u \qquad L = -\partial_x^2 + 4u - 2\partial_x^{-1} u_x \qquad \partial_x^{-1}\partial_x = \partial_x \partial_x^{-1} = 1. \tag{2.3}$$

The well known KdV equation reads

$$u_t - 6uu_x + u_{xxx} = 0. \tag{2.4}$$

For the KdV equation (2.4), the time evolution equation of ϕ is given by

$$\phi_t = 4\lambda\phi_x + 2u\phi_x - u_x\phi. \tag{2.5}$$

The compatibility condition of (2.1) and (2.5) gives rise to (2.4).

It is known that

$$\frac{\delta\lambda}{\delta u} = \phi^2. \tag{2.6}$$

The constrained flows of the KdV hierarchy consists of the equations obtained from the spectral problem (2.1) for N distinct real numbers λ_j and the restriction of the variational derivatives for the conserved quantities H_{k_0} (for any fixed k_0) and λ_j [2–4]

$$-\phi_{j,xx} + u\phi_j = \lambda_j \phi_j \qquad j = 1, \ldots, N \tag{2.7a}$$

$$\frac{\delta H_{k_0}}{\delta u} - \sum_{j=1}^{N} \alpha_j \frac{\delta \lambda_j}{\delta u} = 0. \tag{2.7b}$$

The system (2.7) is invariant under all the KdV flows (2.2).

For $k_0 = 0$, in order to obtain an N-soliton solution, we take

$$\lambda_j < 0 \qquad \zeta_j = \sqrt{-\lambda_j} \qquad \alpha_j = 4\zeta_j \qquad j = 1, \ldots, N$$

one obtains from (2.7b)

$$u = 4\sum_{j=1}^{N} \zeta_j \phi_j^2 = 4\Phi^T \Theta \Phi \tag{2.8}$$

where

$$\Phi = (\phi_1, \ldots, \phi_N)^T \qquad \Theta = \mathrm{diag}(\zeta_1, \ldots, \zeta_N) \qquad \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N).$$

By substituting (2.8), equation (2.7a) becomes

$$-\phi_{j,xx} + 4\sum_{i=1}^{N} \zeta_i \phi_i^2 \phi_j = \lambda_j \phi_j \qquad j = 1, \ldots, N$$

or equivalently

$$\Phi_{xx} = -\Lambda\Phi + 4\Phi\Phi^T\Theta\Phi. \tag{2.9}$$

After inserting (2.8), equation (2.5) reads

$$\Phi_t = 4\Lambda\Phi_x + 8\Phi_x\Phi^T\Theta\Phi - 8\Phi\Phi^T\Theta\Phi_x. \tag{2.10}$$

The compatibility of (2.7), (2.10) and (2.4) ensures that if Φ satisfies two compatible systems (2.9) and (2.10), simultaneously, then u given by (2.8) is a solution of the KdV equation (2.4), namely, the KdV equation (2.4) is factorized by the x-constrained flow (2.9) and the t-constrained flow (2.10).

The Lax representation for the constrained flows (2.9) and (2.10), which can be deduced from the adjoint representation of the spectral problem (2.1) by using the method in [3, 4], is given by

$$Q_x = [\widetilde{U}, Q]$$

where $\widetilde{U}$ and the Lax matrix Q are of the form

$$\widetilde{U} = \begin{pmatrix} 0 & 1 \\ -\lambda + 4\Phi^T\Theta\Phi & 0 \end{pmatrix} \qquad M = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix}$$

$$A(\lambda) = -2\sum_{j=1}^{N} \frac{\zeta_j\phi_j\phi_{j,x}}{\lambda - \lambda_j} \qquad B(\lambda) = 1 + 2\sum_{j=1}^{N} \frac{\zeta_j\phi_j^2}{\lambda - \lambda_j}$$

$$C(\lambda) = -\lambda + 2\Phi^T\Theta\Phi - 2\sum_{j=1}^{N} \frac{\zeta_j\phi_{j,x}^2}{\lambda - \lambda_j}.$$

Then $\frac{1}{2}\,\mathrm{Tr}\, M^2(\lambda) = A^2(\lambda) + B(\lambda)C(\lambda)$, which is a generating function of integrals of motion for the system (2.9) and (2.10), gives rise to

$$A^2(\lambda) + B(\lambda)C(\lambda) = -\lambda - 2\sum_{j=1}^{N} \frac{F_j}{\lambda - \lambda_j}$$

where F_j, $j = 1, \ldots, N$, are N independent integrals of motion for the systems (2.9) and (2.10):

$$F_j = \phi_{j,x}^2 + \left(\lambda_j - 2\sum_{i=1}^{N}\zeta_i\phi_i^2\right)\phi_j^2 + 2\sum_{k\neq j} \frac{\zeta_k(\phi_{j,x}\phi_k - \phi_j\phi_{k,x})^2}{\lambda_j - \lambda_k} \qquad j = 1, \ldots, N. \quad (2.11)$$

3. Deriving the N-soliton solution

In order to construct the N-soliton solution, we have to set $F_j = 0$. It follows from (2.9) that

$$\frac{\phi_{j,x}\phi_k - \phi_j\phi_{k,x}}{\lambda_j - \lambda_k} = -\partial_x^{-1}(\phi_j\phi_k). \quad (3.1)$$

Then one finds

$$F_j = \phi_{j,x}^2 + \left(\lambda_j - 2\sum_{i=1}^{N}\zeta_i\phi_i^2\right)\phi_j^2 - 2\sum_{k=1}^{N}\zeta_k(\phi_{j,x}\phi_k - \phi_j\phi_{k,x})\partial_x^{-1}(\phi_j\phi_k) = 0$$
$$j = 1, \ldots, N. \quad (3.2)$$

The integrals of motion F_j can be used to reduce the order of system (2.9). By multiplying (2.9) by ϕ_j and adding it to (3.2), one obtains

$$-\phi_j\left[\phi_{j,x} - 2\sum_{k=1}^{N}\zeta_k\phi_k\partial_x^{-1}(\phi_j\phi_k)\right]_x + \phi_{j,x}\left[\phi_{j,x} - 2\sum_{k=1}^{N}\zeta_k\phi_k\partial_x^{-1}(\phi_j\phi_k)\right] = 0$$
$$j = 1, \ldots, N$$

which results in

$$\phi_{j,x} - 2\sum_{k=1}^{N}\zeta_k\phi_k\partial_x^{-1}(\phi_j\phi_k) = -\gamma_j\phi_j \qquad \gamma_j = \gamma_j(t) \quad j = 1, \ldots, N$$

or equivalently

$$\Phi_x = -\Gamma\Phi + 2\partial_x^{-1}(\Phi\Phi^T)\Theta\Phi \tag{3.3}$$

where $\Gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_N)$. Set

$$R = 2\partial_x^{-1}(\Phi\Phi^T)\Theta. \tag{3.4}$$

Equation (3.3) can be rewritten as

$$\Phi_x = -\Gamma\Phi + R\Phi. \tag{3.5}$$

Notice that

$$2\Phi\Phi^T = R_x\Theta^{-1} \qquad \Theta R = R^T\Theta \tag{3.6}$$

it follows from (3.4) and (3.5) that

$$\begin{aligned} R_x &= 2\partial_x^{-1}(\Phi_x\Phi^T + \Phi\Phi_x^T)\Theta \\ &= 2\partial_x^{-1}(-\Gamma R_x + RR_x - R_x\Gamma + R_x R) = -\Gamma R - R\Gamma + R^2. \end{aligned} \tag{3.7}$$

We now show that

$$\gamma_j^2 = -\lambda_j \qquad \text{or} \qquad \Gamma^2 = -\Lambda. \tag{3.8}$$

In fact, it is found from (3.5)–(3.7) that

$$\begin{aligned} \Phi_{xx} &= -\Gamma\Phi_x + R\Phi_x + R_x\Phi = \Gamma^2\Phi + (-\Gamma R - R\Gamma + R^2)\Phi + R_x\Phi \\ &= \Gamma^2\Phi + 2R_x\Phi = \Gamma^2\Phi + 4\Phi\Phi^T\Theta\Phi \end{aligned}$$

which together with (2.9) leads to (3.8). Therefore, we can take $\Gamma = \Theta$, equations (3.5) and (3.7) can be rewritten as

$$\Phi_x = -\Theta\Phi + R\Phi \tag{3.9}$$

and

$$R_x = -\Theta R - R\Theta + R^2 \tag{3.10}$$

$$2\Phi\Phi^T = R_x\Theta^{-1} = -\Theta R\Theta^{-1} - R + R^2\Theta^{-1}. \tag{3.11}$$

To solve (3.9), we first consider the linear system

$$\Psi_x = -\Theta\Psi. \tag{3.12}$$

It is easy to see that

$$\Psi = (c_1(t)\,\mathrm{e}^{-\zeta_1 x}, \ldots, c_N(t)\,\mathrm{e}^{-\zeta_N x})^T. \tag{3.13}$$

Take the solution of (3.9) to be of the form

$$\Phi = \Psi - M\Psi$$

then M has to satisfy

$$M_x = -\Theta M + M\Theta - R + RM. \tag{3.14}$$

Comparing (3.14) with (3.10), one finds

$$M = \tfrac{1}{2}R\Theta^{-1} = \partial_x^{-1}(\Phi\Phi^T). \tag{3.15}$$

So we have

$$\Phi = (I - M)\Psi = [I - \partial_x^{-1}(\Phi\Phi^T)]\Psi \tag{3.16}$$

which leads to

$$\Psi = \sum_{n=0}^{\infty} M^n \Phi. \tag{3.17}$$

By using (3.15) and (3.17), it is found that

$$\begin{aligned}\partial_x^{-1}(\Psi\Psi^T) &= \partial_x^{-1}\sum_{n=0}^{\infty}\sum_{l=0}^{n} M^l \Phi\Phi^T M^{n-l} \\ &= \partial_x^{-1}\sum_{n=0}^{\infty}\sum_{l=0}^{n} M^l M_x M^{n-l} = \sum_{n=1}^{\infty} M^n.\end{aligned} \tag{3.18}$$

Set

$$V = (v_{ij}) = \partial_x^{-1}(\Psi\Psi^T) \qquad v_{ij} = -\frac{c_i(t)c_j(t)}{\zeta_i + \zeta_j}\mathrm{e}^{-(\zeta_i+\zeta_j)x}. \tag{3.19}$$

One obtains

$$(I + V)\Phi = \Psi \qquad \text{or} \qquad \Phi = (I - M)\Psi = (I + V)^{-1}\Psi. \tag{3.20}$$

By inserting (3.9) and (3.11), equation (2.10) becomes

$$\begin{aligned}\Phi_t &= [4\Theta^3 - 4\Theta^2 R + 8(-\Theta + R)\Phi\Phi^T\Theta - 8\Phi\Phi^T\Theta(-\Theta + R)]\Phi \\ &= 4\Theta^3\Phi - 4R\Theta^2\Phi.\end{aligned} \tag{3.21}$$

Let Ψ satisfy the linear system

$$\Psi_t = 4\Theta^3\Psi \tag{3.22}$$

then

$$\Psi = (c_1(t)\,\mathrm{e}^{-\zeta_1 x}, \ldots, c_N(t)\,\mathrm{e}^{-\zeta_N x})^T \qquad c_i(t) = \beta_j \mathrm{e}^{4\zeta_j^3 t} \qquad j = 1, \ldots, N. \tag{3.23}$$

We now show that Φ determined by (3.20) and (3.23) satisfy (3.21). In fact, we have

$$\begin{aligned}\Phi_t &= -(I + V)^{-1}V_t(I + V)^{-1}\Psi + (I + V)^{-1}\Psi_t \\ &= 4\Theta^3\Phi - 4M\Theta^3\Phi - 4(I - M)V\Theta^3\Phi \\ &= 4\Theta^3\Phi - 8M\Theta^3\Phi = 4\Theta^3\Phi - 4R\Theta^2\Phi.\end{aligned}$$

Therefore, Φ given by (3.20) and (3.23) satisfies (2.9) and (2.10) simultaneously and $u = 4\Phi^T\Theta\Phi$ is the solution of KdV equation (2.4). It is easy to show that this solution is just the N-soliton solution. Notice that

$$\begin{aligned}2\partial_x(\Psi^T\Phi) &= -2\Psi^T\Theta\Phi + 2\Psi^T(-\Theta + R)\Phi \\ &= -4\Phi^T(I + V)(I - M)\Theta\Phi = -4\Phi^T\Theta\Phi\end{aligned}$$

namely

$$u = -2\partial_x \sum_{i=1}^{N} c_i(t)\,\mathrm{e}^{-\zeta_i x}\phi_i. \tag{3.24}$$

Formulae (3.20), (3.23) and (3.24) are just those obtained from the Gel'fand–Levintan–Marchenko equation for determining the N-soliton solution for the KdV equation [17–19] and finally results in the well known expression for the N-soliton solution of the KdV equation (2.4)

$$u = -2\partial_x^2 \ln(\det(I + V)).$$

4. Conclusion

The factorization of the KdV equation into two compatible x- and t-constrained flows enables us to derive directly the N-soliton solution via the x- and t-constrained flows. The method presented here can be applied to other soliton equations for directly obtaining N-soliton solutions from constrained flows.

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